Topological Data Analysis

A Primer on Topology

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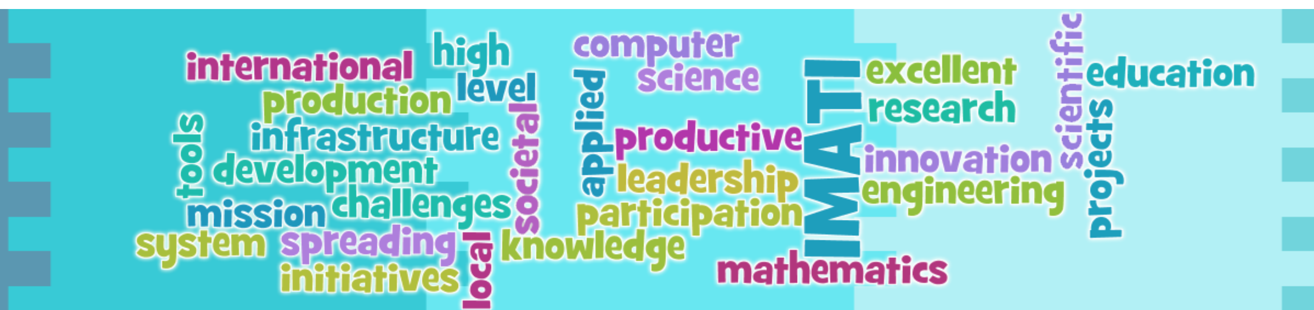
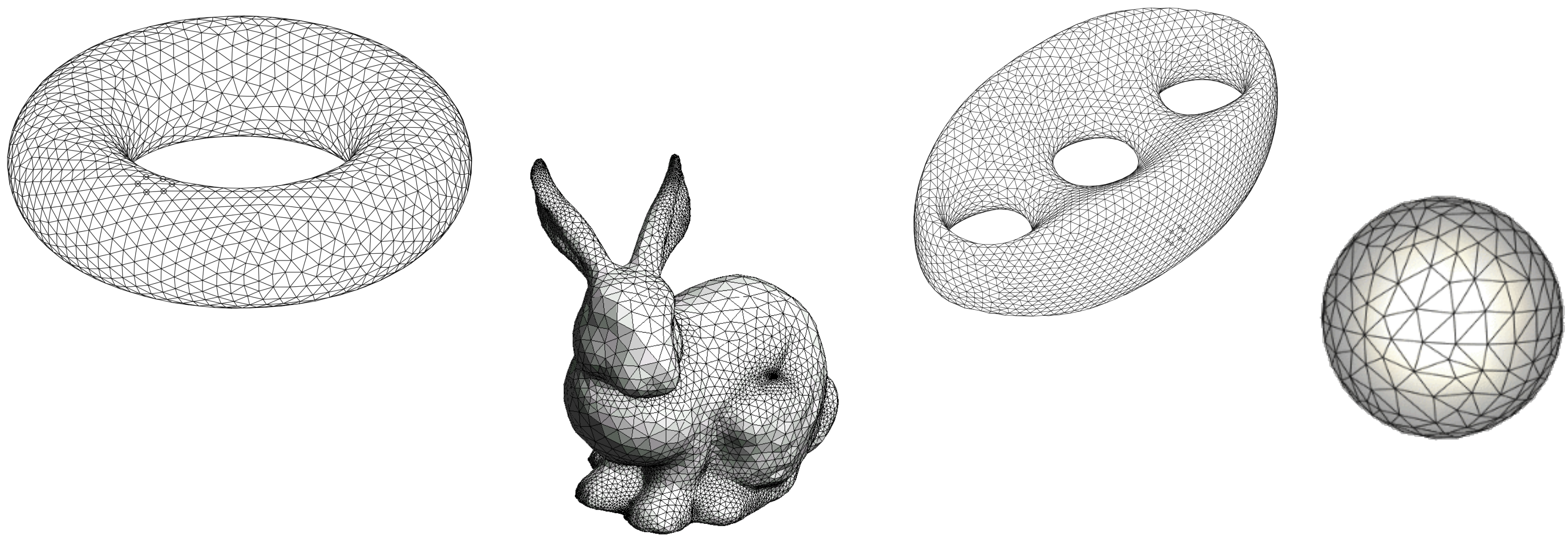
CNR - IMATI



Topological Data Analysis

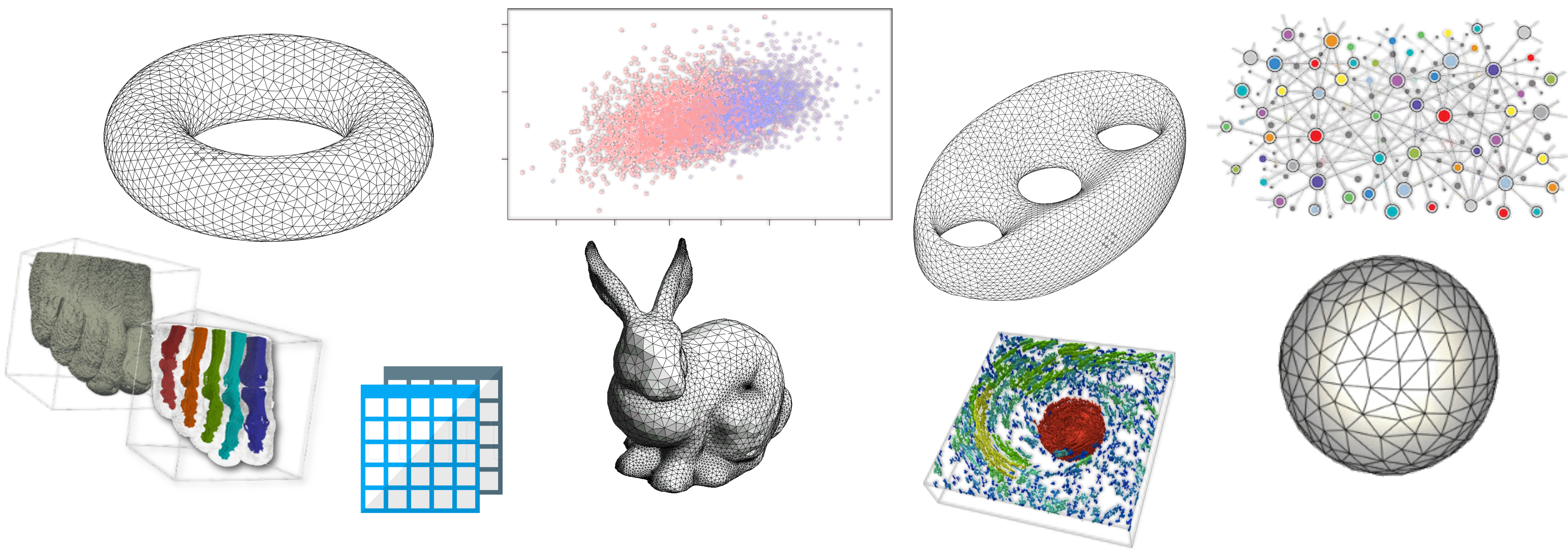
Topology *describes*, *characterizes,* and *discriminates* ***shapes*** by studying their properties that are preserved under ***continuous deformations***, such as ***stretching*** and ***bending***, but ***not tearing*** or ***gluing***



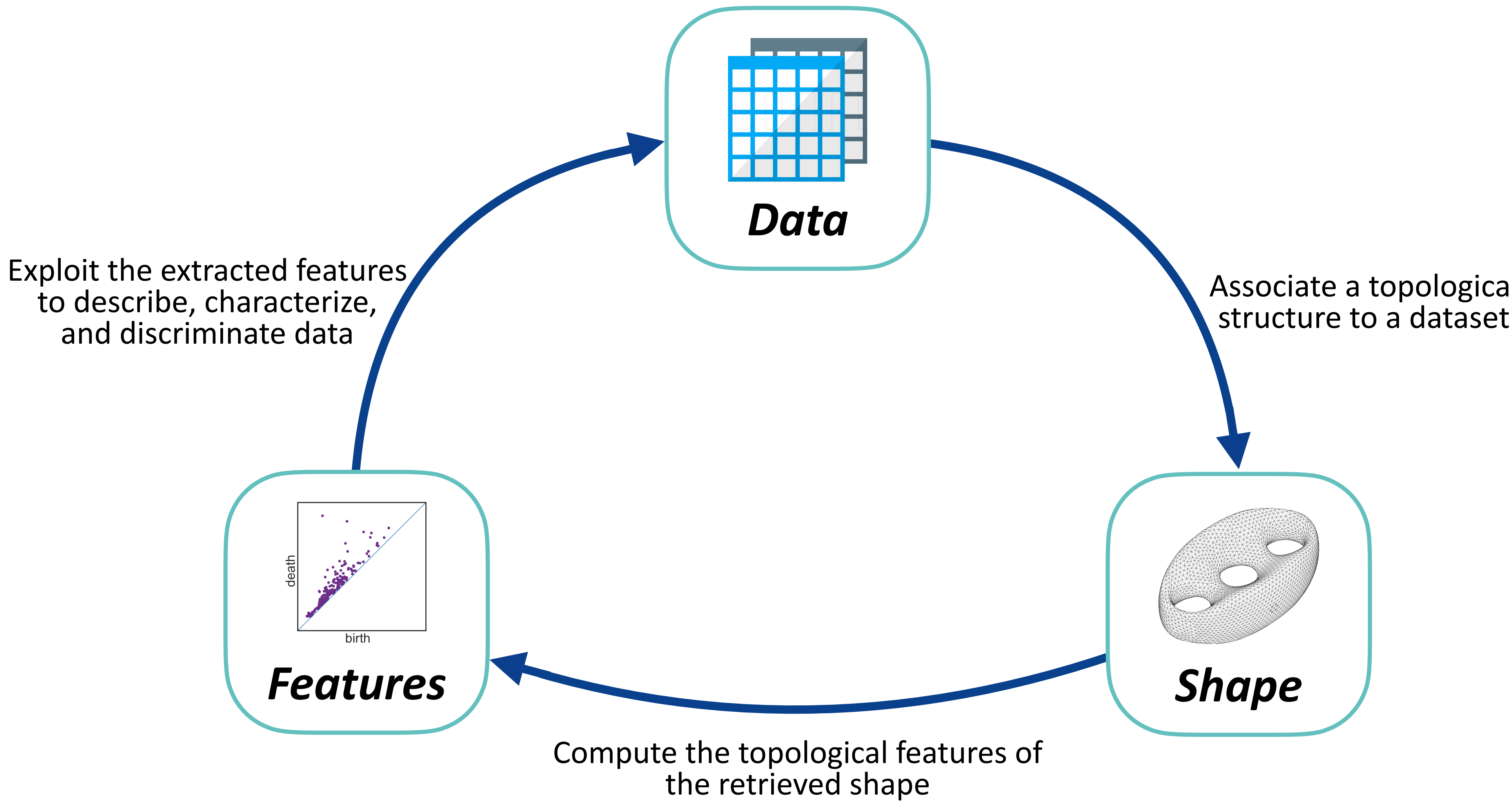
Topological Data Analysis

Assumption in TDA: ***Any data*** can be endowed with a ***shape***.

So, any data can be studied in terms of its ***topological features***



Topological Data Analysis

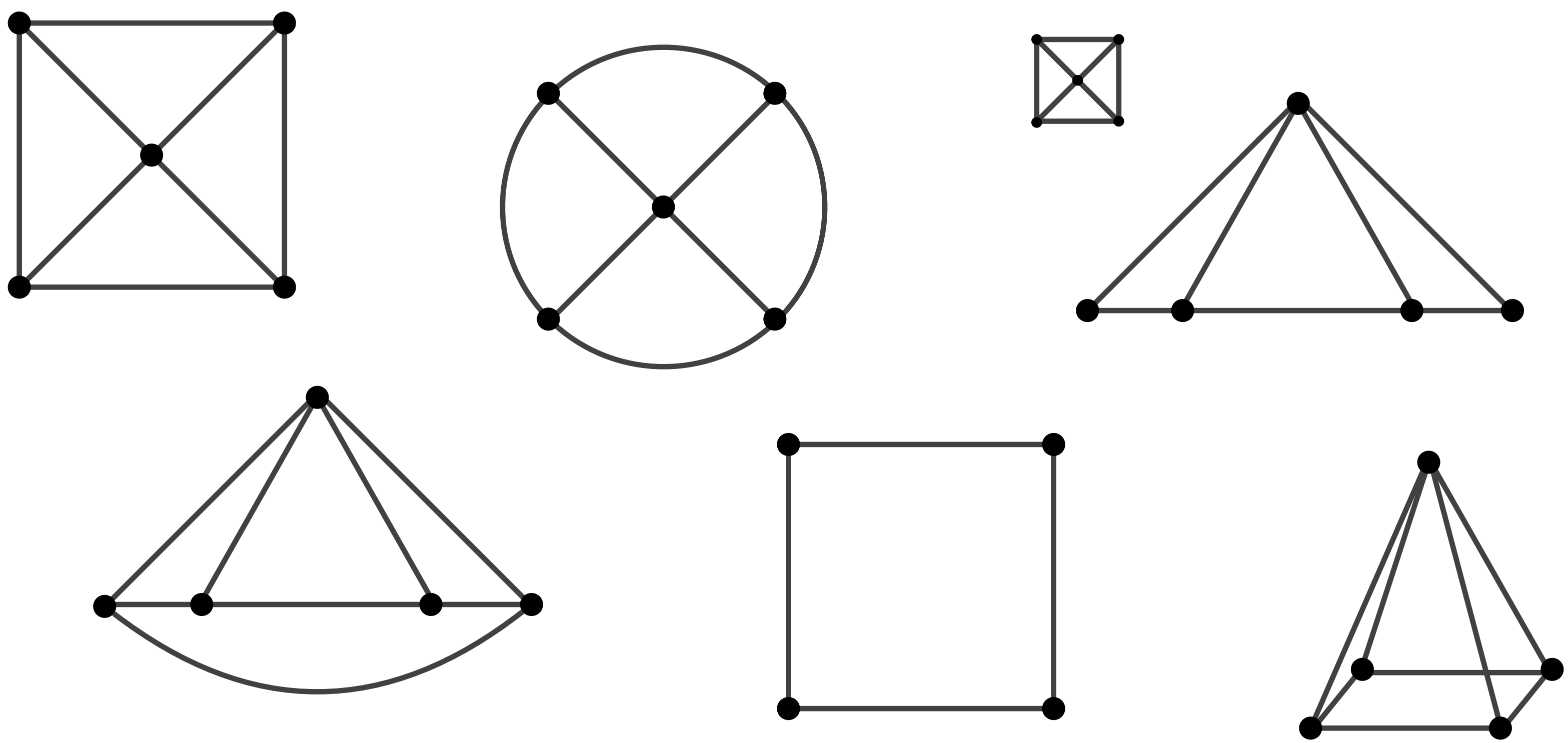


Geometry or Topology?

Which of these domains look similar?

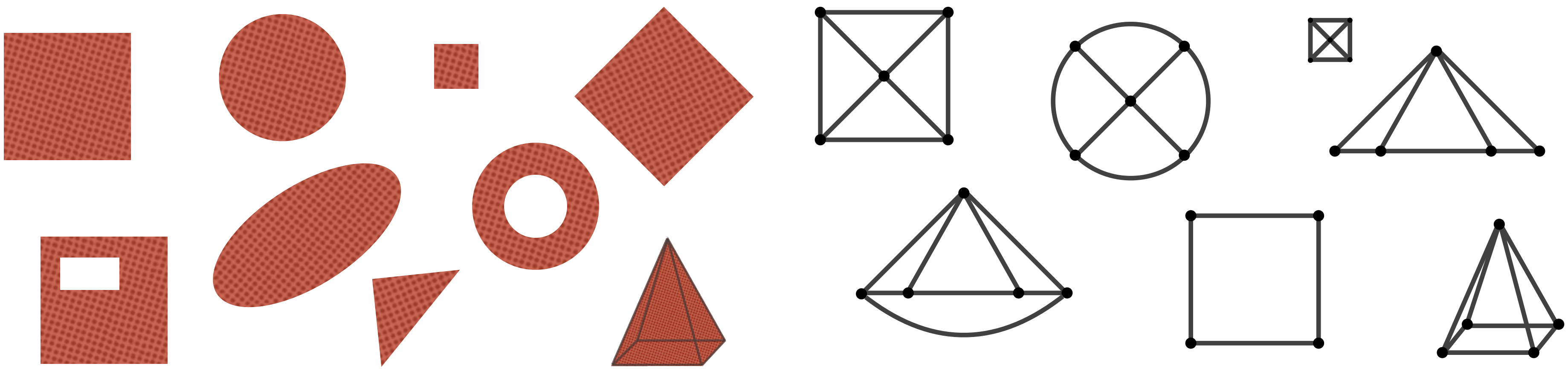
Geometry or Topology?

And what about these ones?



Geometry or Topology?

The answer depends on the *point of view* we adopt



*Geometry cares about those properties which **change** when an object is continuously **deformed***

E.g. length, area, volume, angles, curvature, …

Geometry or Topology?

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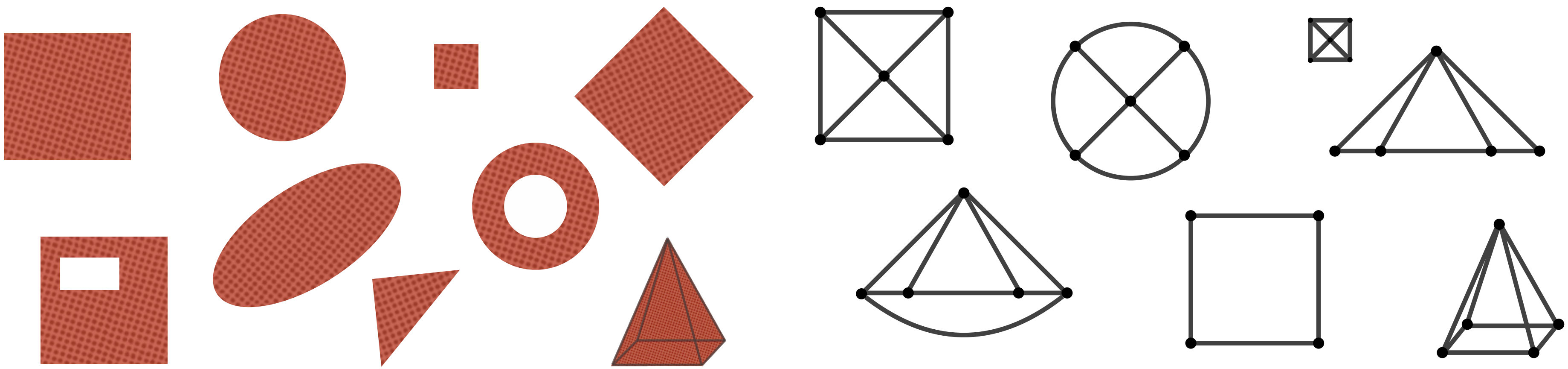


Topology *do not*

*~~Geometry~~ cares about those properties which **change***

*when an object is continuously **deformed***

E.g. connectivity, orientation, manifoldness, …

Why Topology?

In life or social sciences, **distances** (**metric**) are constructed using a notion of **similarity** (**proximity**): e.g. distance between faces, gene expression proles, Jukes-Cantor distance between sequences

We have that:

- Construction of a distance has ***no theoretical backing***
- Small distances still represent similarity, but ***long distance comparisons hardly make sense***
- Distance measurements are ***typically noisy***
- Physical devices, e.g. human eyes, may ***ignore differences in proximity***

Topology is the crudest way to capture invariants under distortions of distances

(even if, at the presence of noise, one needs topology varied with scales)

Topological Spaces

Definition:

A ***topological space (X, T)*** is a non-empty set X endowed with a family T, called ***topology***, of subsets of X satisfying the following properties:

- ***X*** *and the* ***empty set*** *∅ belong to T*
- ***Union of any collection*** *of elements of T is in T*
- ***Intersection of any finite collection*** *of elements of T is in T*

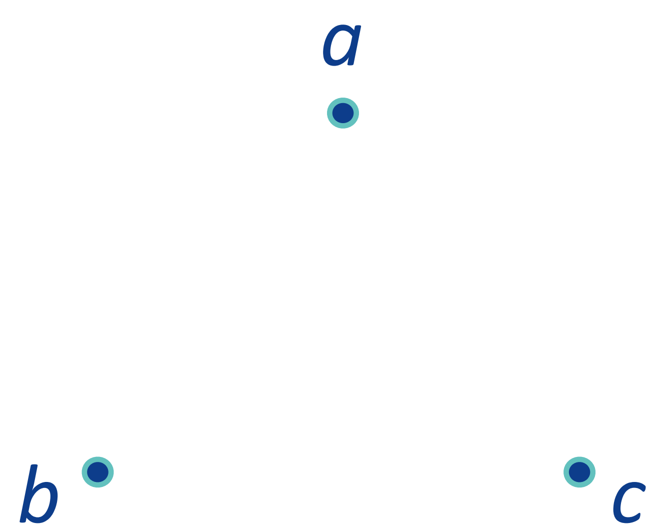
A set U in T is called ***open set***. A set F such that X \ F is in T is called ***closed set***

Dually to the above definition, a topological space can be characterized by defining its closed sets

Topological Spaces

Exercise:

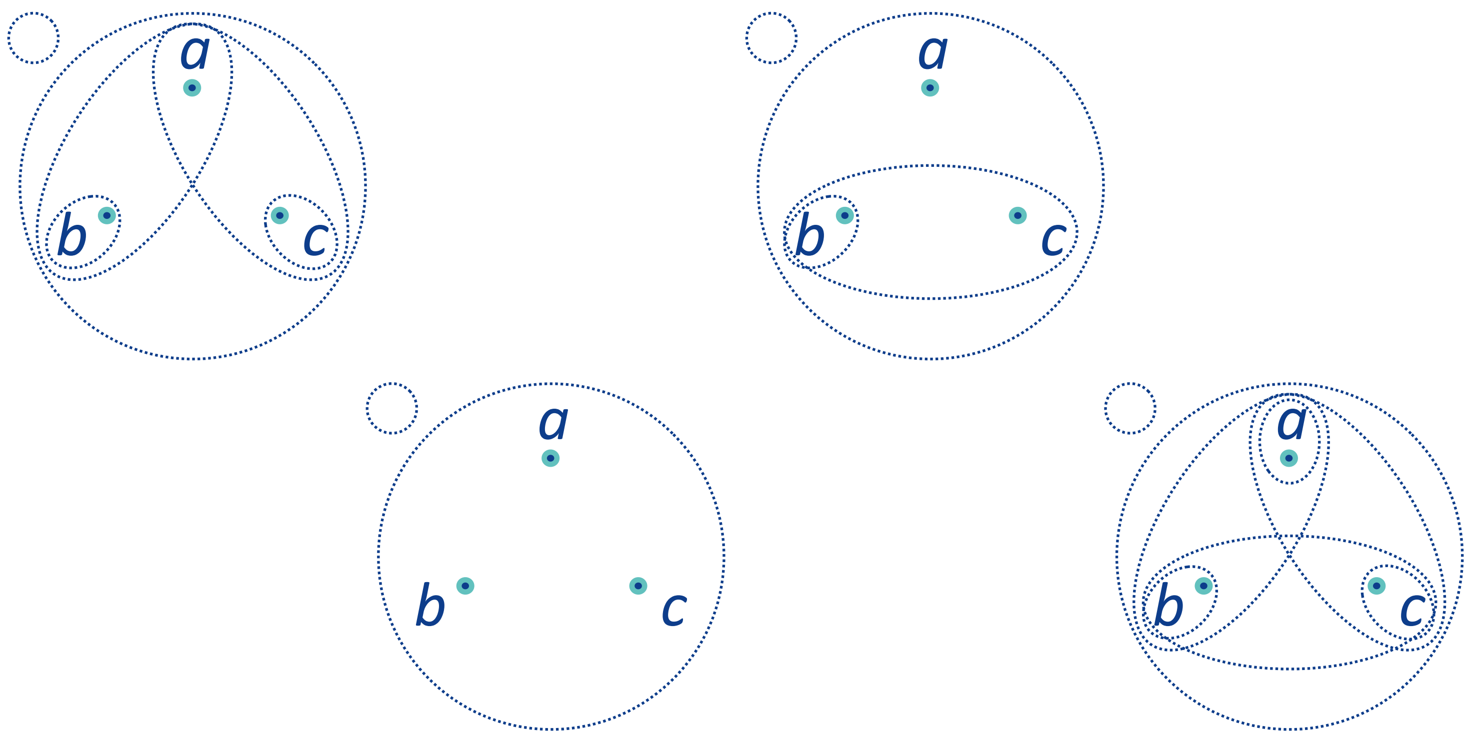
Given the set $X := \{a, b, c\}$, define a topology T for X



Topological Spaces

Exercise:

Which of the following families are topologies for X?



Topological Spaces

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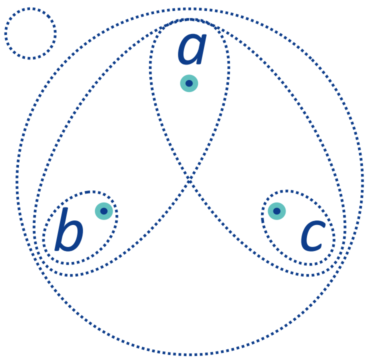
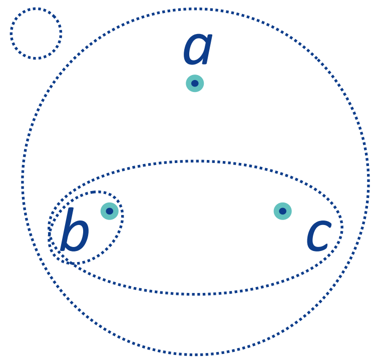
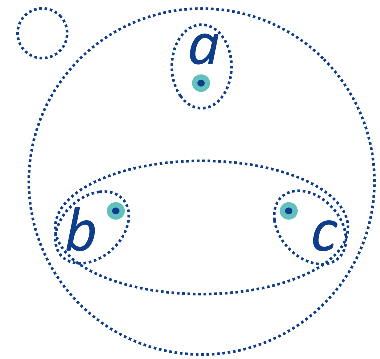
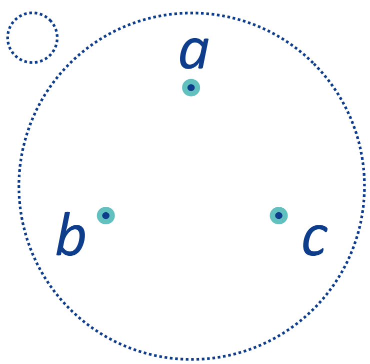
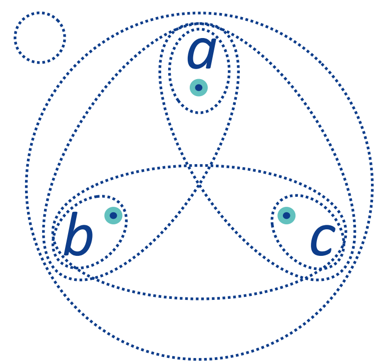






***Trivial* topology:**
i.e. $T := \{\emptyset, X\}$





***Discrete* topology:**
i.e. $T := P(X)$

Topological Spaces

Definition:

Let T be a topology of a non-empty set X. A ***basis*** of T is a family of open sets $\mathscr{B} \subseteq$ T such that ***each open sets of T is union of elements of*** $\mathscr{B}$

Proposition:

Let X be a non-empty set and $\mathscr{B}$ be a family of subsets of X such that:

- $\bigcup_{B \in \mathscr{B}} B = X$
- For any A, B $\in \mathscr{B}$, A $\cap$ B is union of elements of $\mathscr{B}$

Then, ***there exists a (unique) topology*** T of X of which $\mathscr{B}$ is a basis

Metric Spaces as Topological Spaces

Definition:

A ***metric space (X, d)*** is a non-empty set X on which is defined a function $d: X \times X \longrightarrow \mathbb{R}$, called ***distance***, such that, for any $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ if and only if $x = y$ *(identity of indiscernibles)*
- $d(x, y) = d(y, x)$ *(symmetry)*
- $d(x, z) \leq d(x, y) + d(y, z)$ *(subadditivity or triangle inequality)*

Proposition:

Each metric space (X, d) is a topological space (X, T) with respect to the topology T having as basis $\mathscr{B} := \{B(x, r) \mid x \in X, r > 0\}$, where B(x, r) is the ***open ball of radius r centered in x*** defined as $B(x, r) := \{y \in X \mid d(x, y) < r\}$

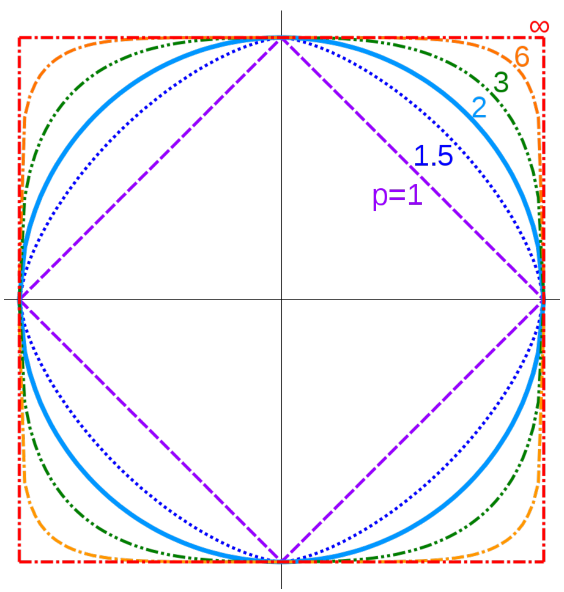
Metric Spaces as Topological Spaces

Example: The ***n-dimensional*** ***Euclidean space*** $\mathbb{E}^n$ is the topological space induced by the metric space $(\mathbb{R}^n, d)$ where d is defined as

$$d(x,y) := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

For any $p \geq 1$, the ***Minkowski distance*** d_p induces the same topology on $\mathbb{R}^n$

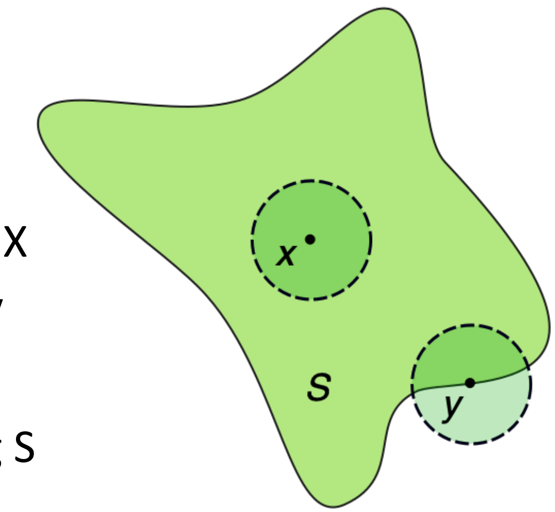
$$d_p(x,y) := \left(\sum_{i=1}^{n}|x_i - y_i|^p\right)^{1/p}$$



Topological Spaces

Some Basic Notions:

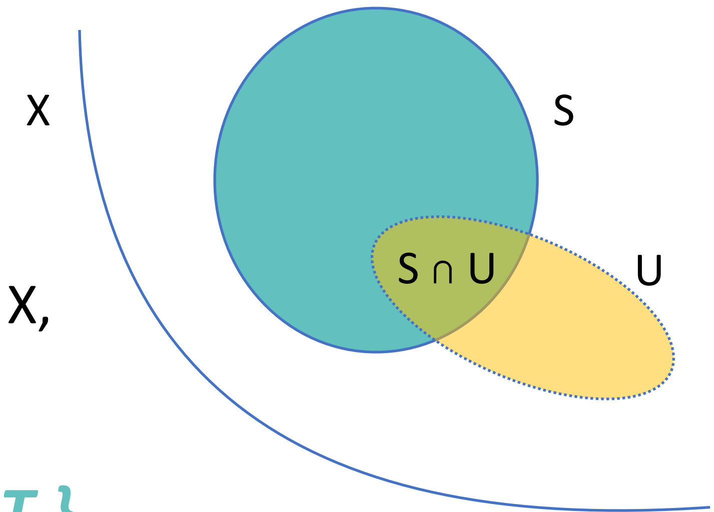
Given a topological space (X, T), an element x of X, and a subset S of X:



- A ***neighborhood of x*** is a subset V of X that includes an open set U containing x (i.e. $x \in U \subseteq V$)
- The ***interior i(S)*** of S is the union of all subset of S that are open of X
  - i(S) consists of the elements x of X for which there exists an open neighborhood V of x completely contained in *S*
- The ***closure c(S)*** of S is the intersection of all closed sets containing S
  - c(S) consists of the elements x of X for which every open neighborhood V of x contains a element of S
- The ***boundary ∂(S)*** of S is the set of elements in the closure of S not belonging to the interior of S (i.e. $\partial(S) = c(S) \setminus i(S)$)
  - ∂(S) consists of the elements x of X for which every open neighborhood V of x intersects both S and X \ S

Topological Spaces

Definition:



Given a topological space (X, T) and a subset S of X, the ***subspace topology T_S*** on S is defined as

$$T_S := \{ S \cap U \mid U \in T \}$$

I.e. a subset of S is an open set of T_S if and only if it is the intersection of S with an open set of X

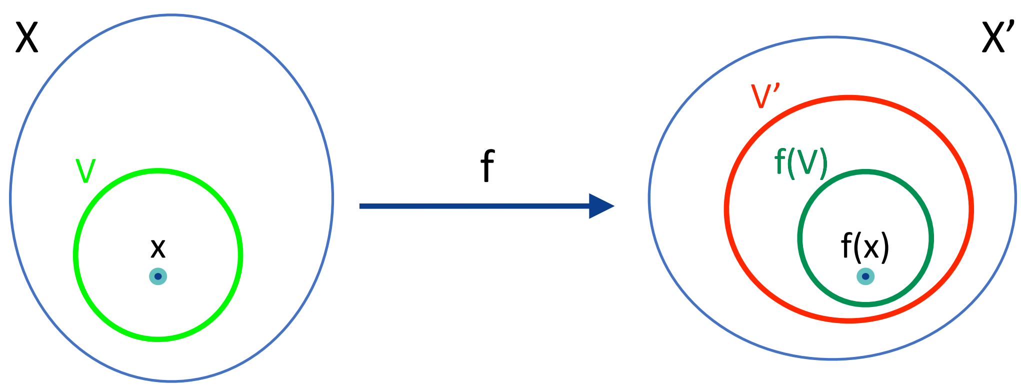
S equipped with the subset topology T_S is called a ***subspace*** of (X, T)

Continuous Functions

Definition:

Given two topological spaces (X, T) and (X', T'), a ***function f: X ⟶ X'*** is called

- ***Continuous in x ∊ X*** if, for each neighborhood V' of f(x) in X', there exists a neighborhood V of x in X such that f(V) ⊆ V'
- ***Continuous*** if it is continuous in each element x ∊ X or, equivalently, if, ***for each open set U' of X', f^{-1} (U') is an open set of X***



Continuous Functions

Exercise:

Let X be a non-empty set X and let T, T' be the discrete and the trivial topologies on X, respectively. Which of the following functions is continuous?

- *the identity map* $\mathbf{id: (X, T) \longrightarrow (X, T')}$
- *the identity map* $\mathbf{id': (X, T') \longrightarrow (X, T)}$

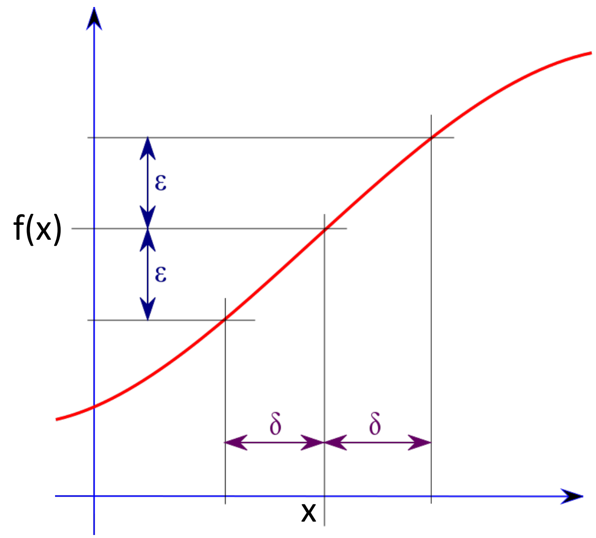
Continuous Functions

Proposition:

Given two metric spaces (X, d) and (X', d'), **a function $f: X \longrightarrow X'$ is continuous in $x \in X$**

if and only if

$\forall \varepsilon > 0\ \exists\ \delta > 0$ such that, for any $y \in X$ with $d(x, y) < \delta$, $d'(f(x), f(y)) < \varepsilon$

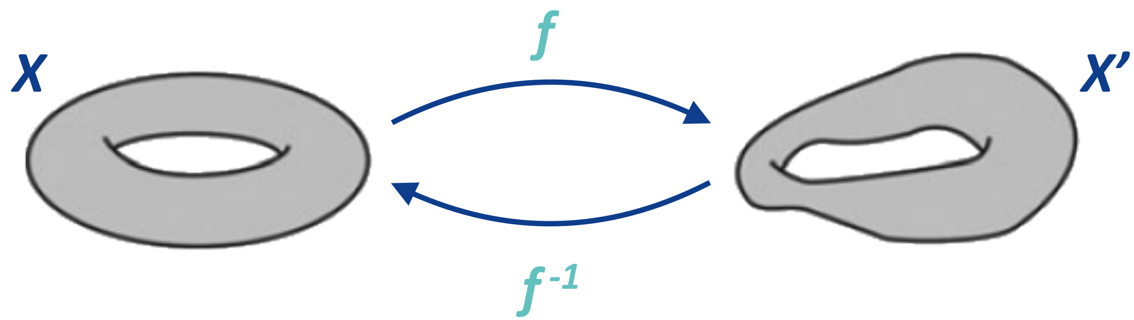


Homeomorphisms

Definition:

Given two topological spaces (X, T) and (X', T'), a function f: X ⟶ X' is called ***homeomorphism*** if:

- f is a ***bijection***
- f is ***continuous***
- f^{-1} is ***continuous***

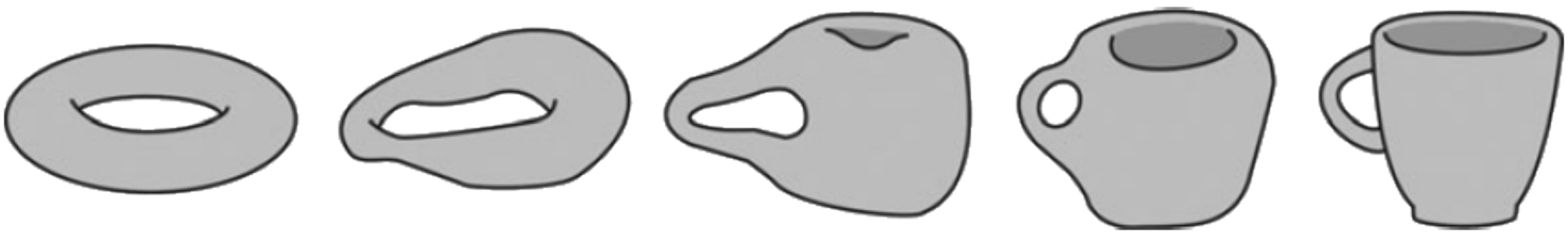


Two topological spaces (X, T) and (X', T') are ***homeomorphic*** and denoted ***$X \cong X'$*** if there exists a homeomorphism f: X ⟶ X'

Homeomorphisms induce an ***equivalence relation*** of topological spaces partitioning them into equivalence classes

Homeomorphisms

Intuitively:



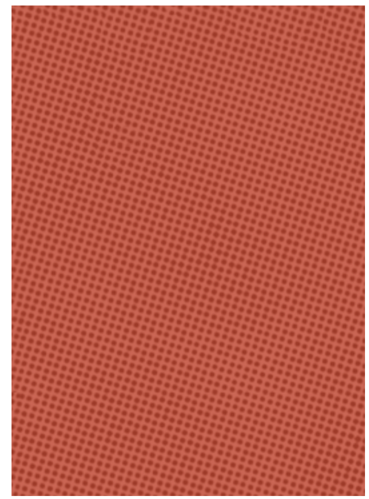
The notion of homeomorphism captures the idea of continuous deformation

$\cong$

Homeomorphisms

Intuitively:

One can:

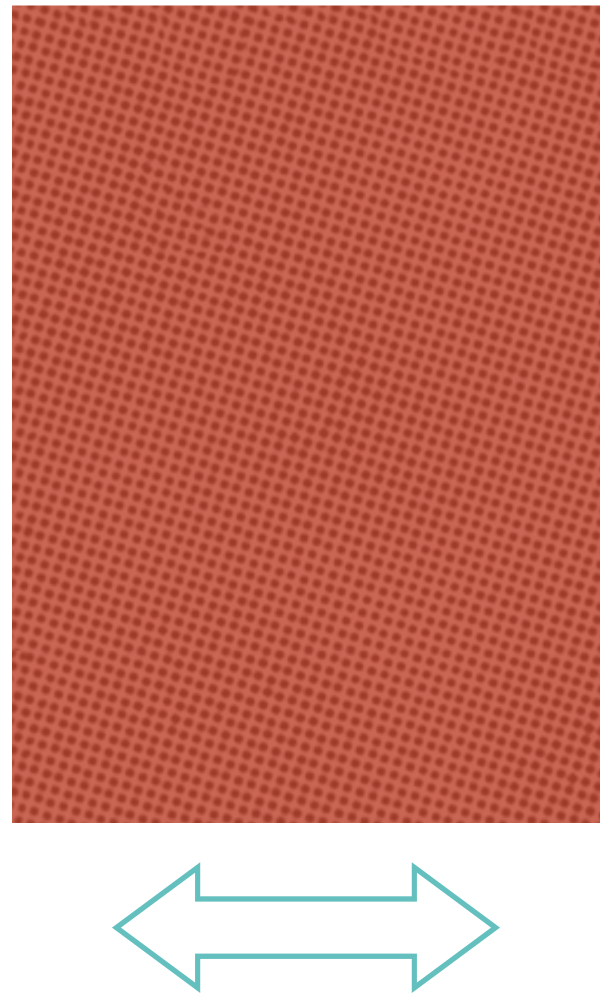


Homeomorphisms

Intuitively:

One can:

- Stretch



Homeomorphisms

Intuitively:

One can:

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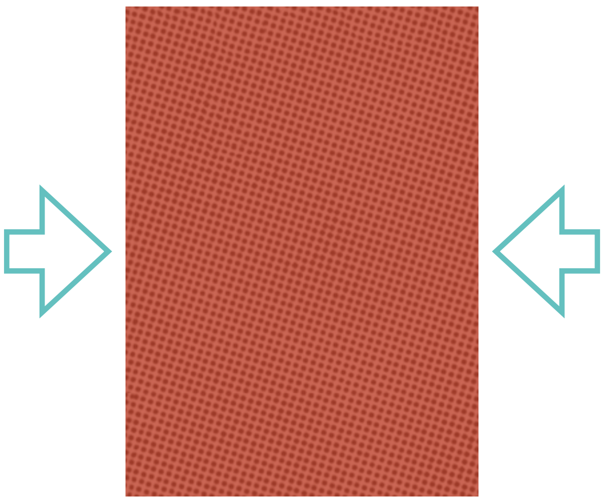
- Compress



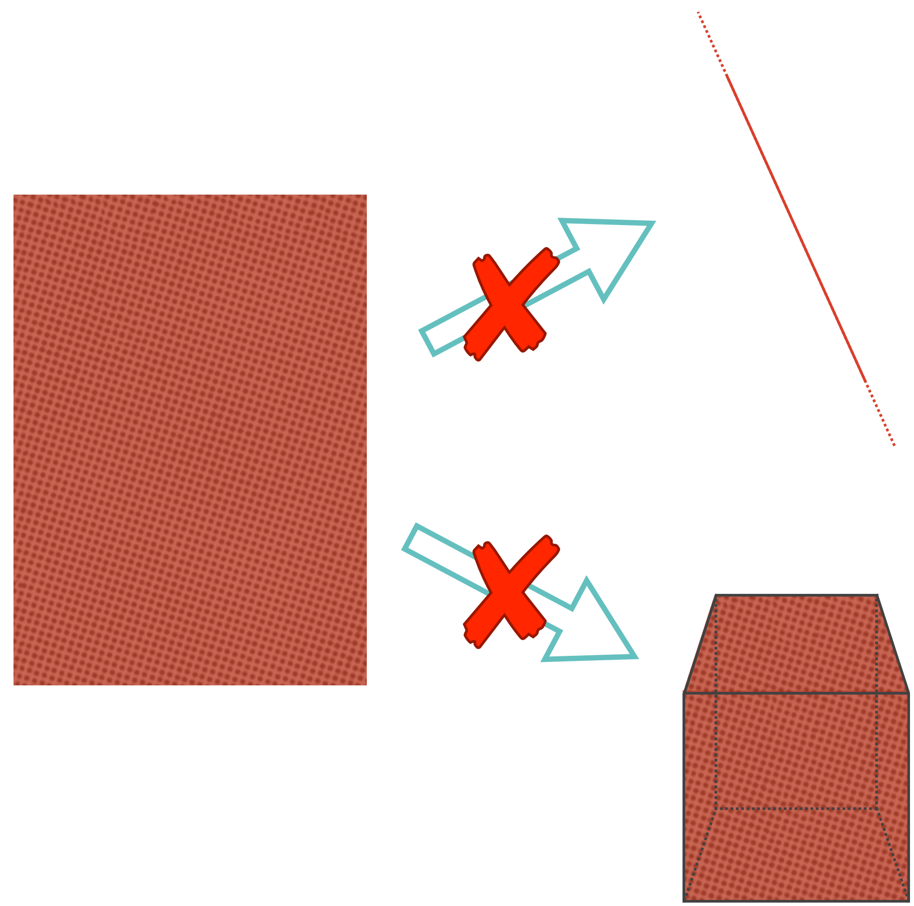
Homeomorphisms

Intuitively:

One can:

- Stretch
- Compress

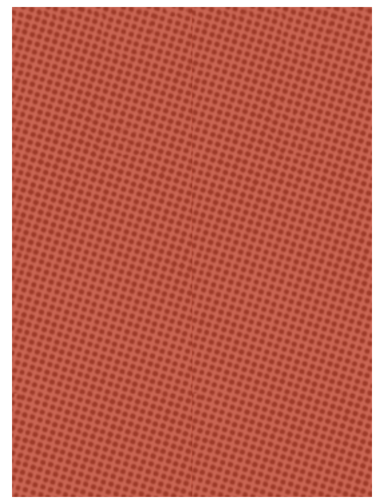
But not too much!



Homeomorphisms

Intuitively:

Moreover:

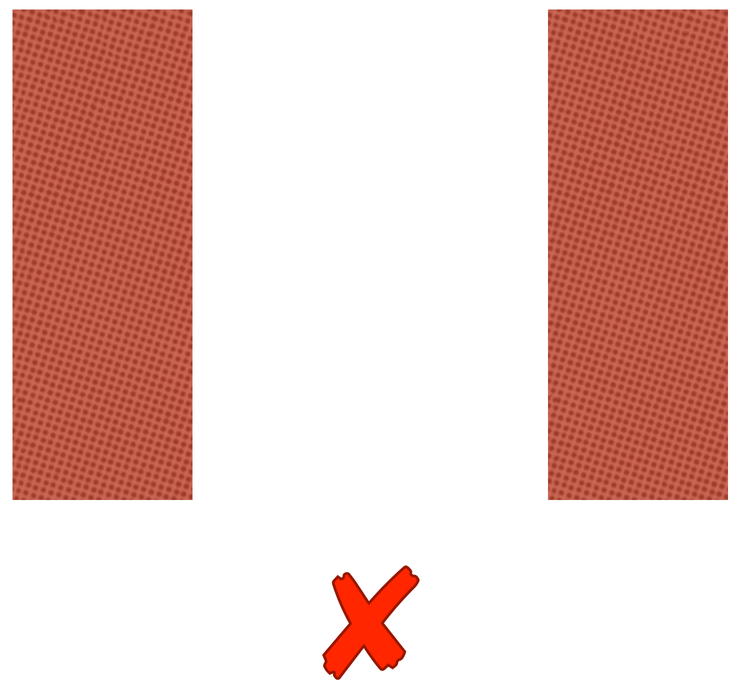


Homeomorphisms

Intuitively:

Moreover:

- No Cut



Homeomorphisms

Intuitively:

Moreover:

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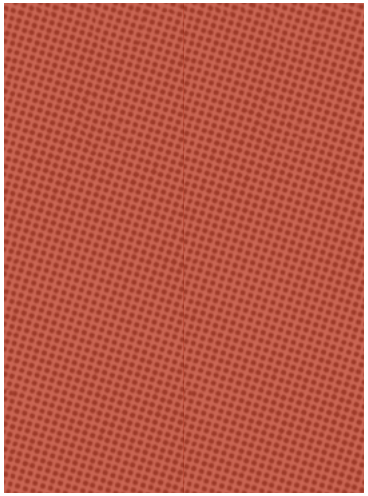
- No Glue

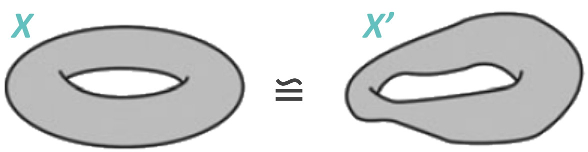


Topological Invariants

Definition:

I is a ***topological invariant*** if, given two topological spaces (X, T) and (X', T'),

X is *homeomorphic* to *X'*



X and *X'* have the *same topological invariant*

I(X) = I(X')

Some classical topological invariants:

- *Connectedness*
- *Compactness*
- *Manifoldness*
- *Orientability*
- *Euler characteristic*
- *Homology*
- *Homotopy*

Topological Invariants

Question:

Is there a "perfect" topological invariant I such that

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Let us **simplify the question** and let focus on:

- *Considering a specific topological invariant I (e.g. the **homology**)*
- *Completely characterizing just the **spheres** $S^n := \{ x \in \mathbb{R}^n : |x| = 1 \}$*

The above question turns into the following:

If X and S^n have the same homology, then $X \cong S^n$?

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NO

Topological Invariants

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Topological Invariants

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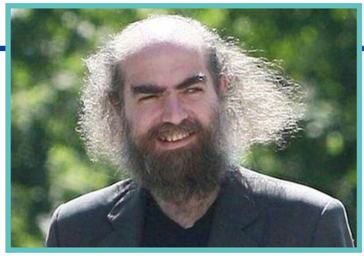
Poincaré Conjecture *(3rd Millennium Prize Problem)**:***

If X is a closed n-manifold homotopy equivalent to S^n, then $X \cong S^n$



Proven by Grigori Perelman in 2003

Topological Invariants

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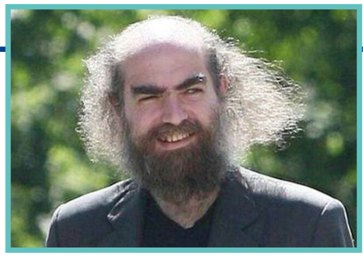

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So: *Why we will mainly focus on homology rather than homotopy?*

Topological Invariants

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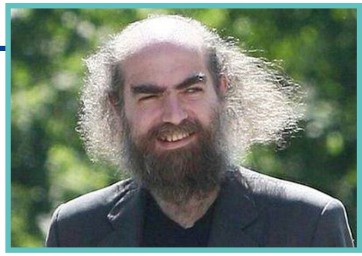


Proven by Grigori Perelman in 2003

So: *Why we will mainly focus on homology rather than homotopy?*

Because, in practice, computing homotopy groups is nearly impossible!

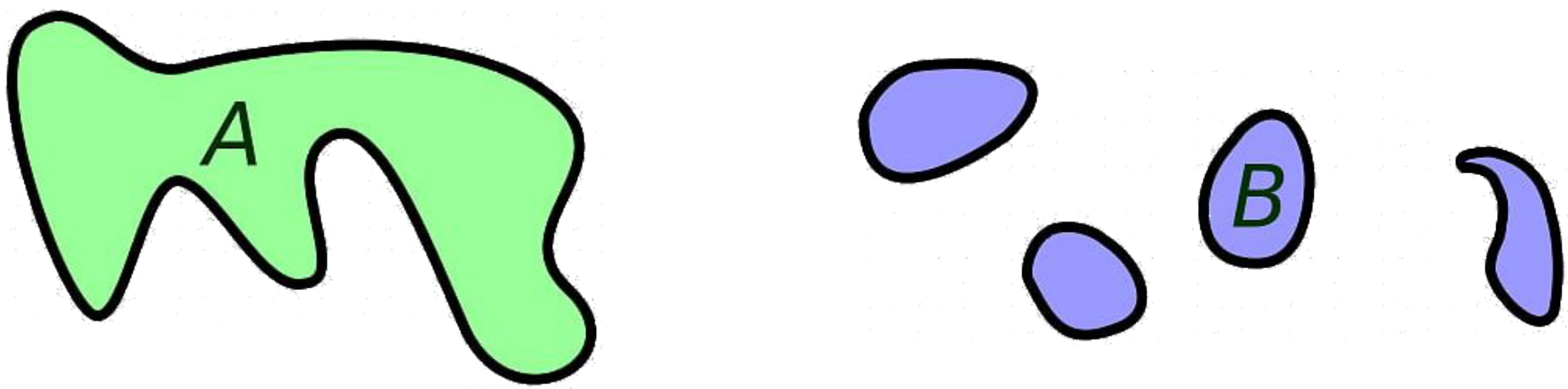
Connected Spaces

Definition:

A topological space (X, T) is ***connected*** if, given any two disjoint open sets U and V s.t.

$$X = U \cup V, \text{ then } U = \varnothing \text{ or } V = \varnothing$$

I.e. X cannot be written as the union of two non-empty disjoint open sets of X



A ***connected component*** of X is a maximal connected subset of X

Connected Spaces

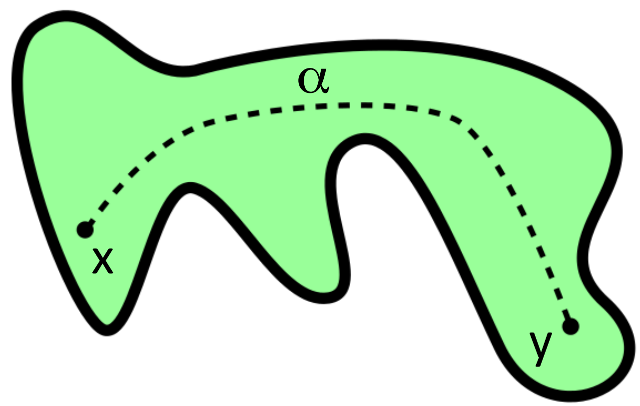
Definition:

A topological space (X, T) is ***path-connected*** if, for every pair x, y $\in$ X, there exists

a ***continuous map*** α***: [0,1]*** $\longrightarrow$ ***X*** such that α***(0) = x*** and α***(1) = y***

The map α is called a ***path*** from x to y

A ***path-connected component*** of X is a maximal path-connected subset of X



Proposition:

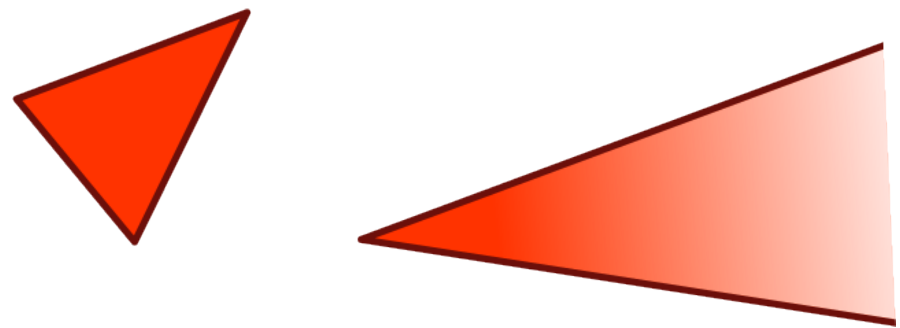
If X is path-connected, then X is connected. The converse is not true

Compact Spaces

Definition:

An ***open cover*** of a topological space (X, T) is a collection C of open sets U_i of X whose union is the whole space X, i.e. $X \supseteq \bigcup_{i \in I} U_i$. A ***subcover*** of C is a subset of C that still covers X

A topological space (X, T) is called ***compact*** if any of its open covers has a ***finite subcover***



Heine-Borel Theorem:

A subset S of the Euclidean space $\mathbb{E}^n$ is ***compact if and only if*** S is ***closed*** and ***bounded***
(i.e. there exists r > 0 such that, for any x, y ∊ S, d(x, y) < r)

Manifolds

Definitions:

A topological space (X, T) is called

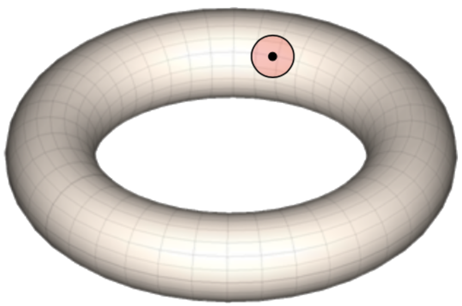
- ***locally homeomorphic*** to $\mathbb{E}^n$ if every element $x \in X$ has a neighborhood which is homeomorphic to the n-dimensional Euclidean space $\mathbb{E}^n$
- ***Hausdorff*** if any pair of distinct elements $x, y \in X$ admits disjoint neighborhoods (any metric space and so any subspace of an Euclidean space is Hausdorff)

A ***(topological) n-manifold*** M is a ***Hausdorff*** space that is ***locally homeomorphic*** to the n-dimensional Euclidean space $\mathbb{E}^n$

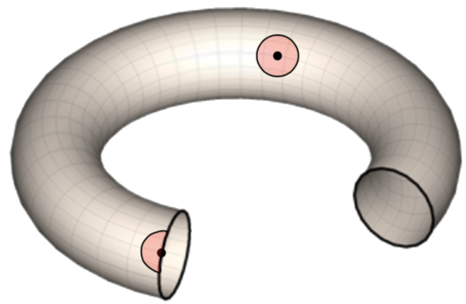
A ***(topological) n-manifold with boundary*** M is a ***Hausdorff*** space in which every element has a ***neighborhood homeomorphic*** to the n-dimensional ***Euclidean space*** $\mathbb{E}^n$ or to the n-dimensional ***Euclidean half-space*** $H^n := \{ x \in \mathbb{R}^n \mid x_n \geq 0 \}$

Manifolds

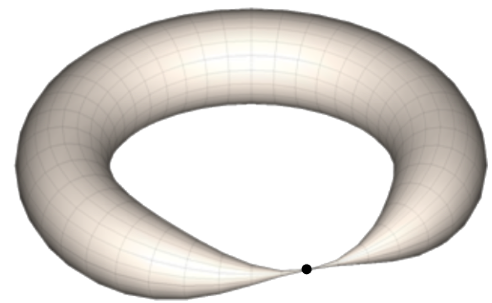
Examples:



manifold

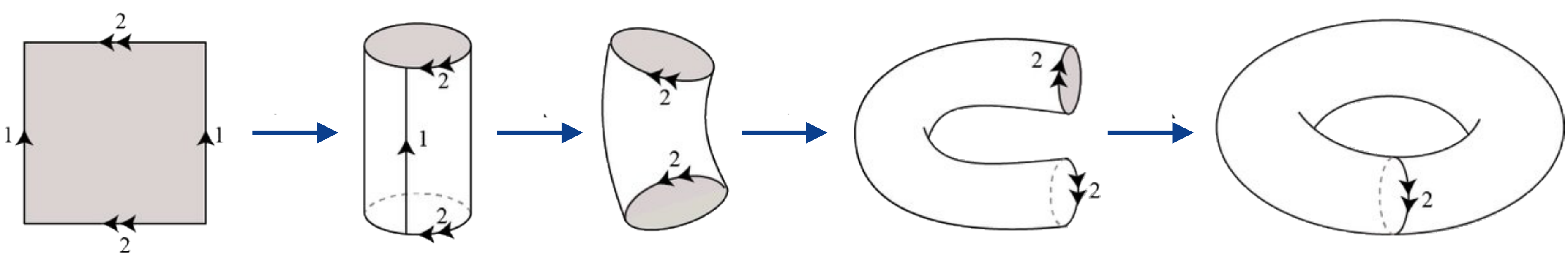


manifold with boundary



non-manifold

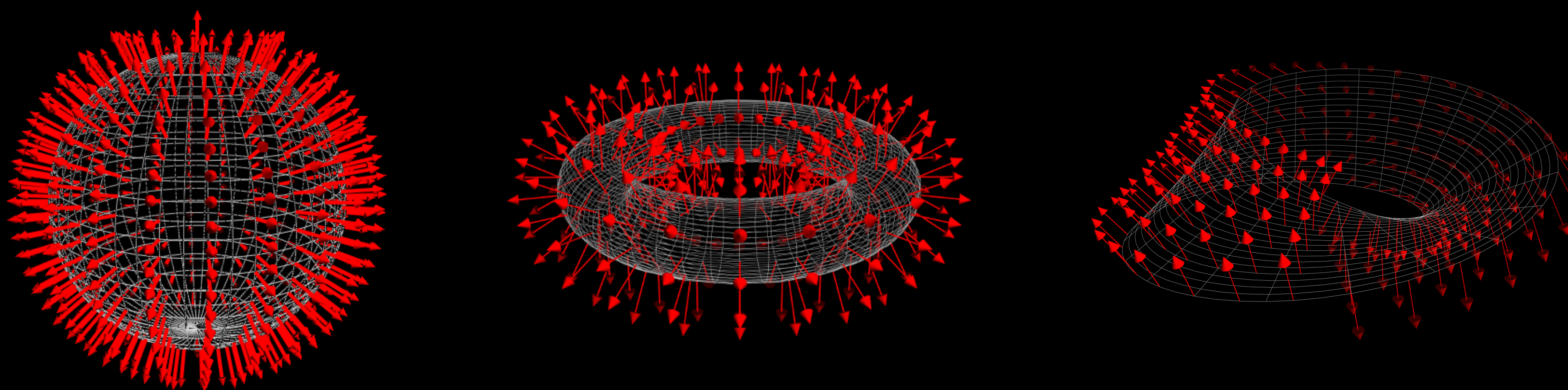
Recall that a ***torus*** can be built from a ***unit square*** by the following construction



Orientable Surfaces

Definition:

A ***surface*** (i.e. a topological 2-manifold with or without boundary) S is called ***orientable*** if it is possible to make a ***consistent choice of surface normal vector*** at every point of S

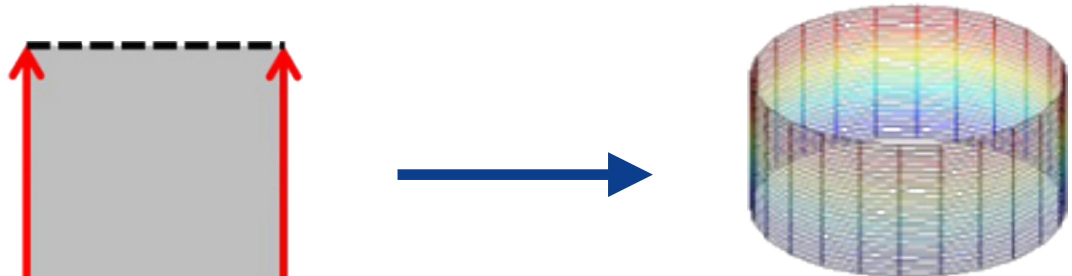


orientable ***orientable*** ***non-orientable***

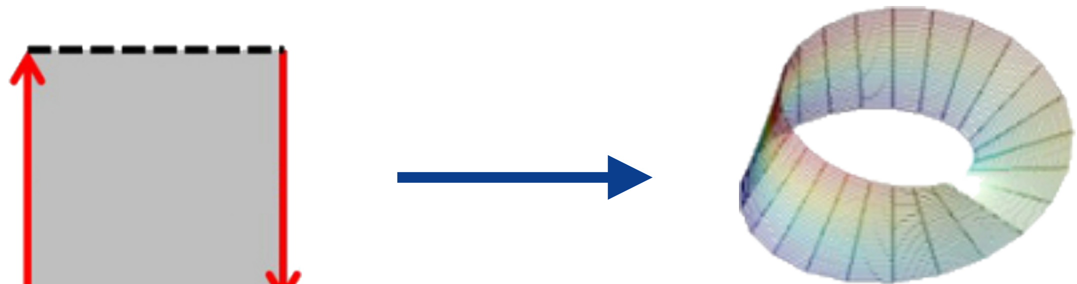
Orientable Surfaces

Remark:

As for the *torus* and the *cylinder*,
the ***Möbius strip*** can be built from a unit square via edge identification



cylinder



Möbius strip

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